Q: Sketch a graph of the function $f(x) = \dfrac{2x^2 - 2}{(x+3)^2}$.

A: First, let's locate a few points on the graph by finding the zeros. $f(x) = 0$ when $2x^2 - 2 = 0$; solving, this is when $x = \pm 1$, so the points $(-1, 0)$ and $(1, 0)$ lie on the graph. Next, let's look for asymptotes. To find the horizontal asymptotes, we note that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^2-2}{(x+3)^2} = \lim_{x\to\infty} \frac{2x^2-2}{x^2+6x+9}$$

$$= \lim_{x\to\infty} \frac{x^2\left(2 - \frac{2}{x^2}\right)}{x^2\left(1 + \frac{6}{x} + \frac{9}{x^2}\right)} = \lim_{x\to\infty} \frac{2 - \frac{2}{x^2}}{1 + \frac{6}{x} + \frac{9}{x^2}}$$

$$= \frac{\lim_{x\to\infty} 2 - \frac{2}{x^2}}{\lim_{x\to\infty} 1 + \frac{6}{x} + \frac{9}{x^2}} = \frac{2}{1} = 2;$$

similarly, we can also verify that $\lim_{x\to-\infty} f(x) = -2$. So we have a horizontal asymptote $y = 2$ to the left and to the right. To check for vertical asymptotes, notice that the function is undefined at $x = -3$, so there's potentially a vertical asymptote there. To be certain, we need to check that

$$\lim_{x\to-3^+} f(x) = \lim_{x\to-3^+} \frac{2x^2-2}{(x+3)^2} = \infty,$$

since $\lim_{x\to-3^+} 2x^2 - 2 = 16$, $\lim_{x\to-3^+} (x+3)^2 = 0$, and $(x+3)^2 > 0$ in a neighborhood to the right of -3, and similarly that

$$\lim_{x\to-3^-} f(x) = \lim_{x\to-3^-} \frac{2x^2-2}{(x+3)^2} = \infty.$$

Therefore $f(x)$ has a vertical asymptote at $x = -3$. Next, we'd like to know about monotonicity (in other words, the property of being increasing or decreasing) and convexity, so we should calculate the first and second derivatives:

$$f'(x) = \frac{(x+3)^2(2x^2-2)' - (2x^2-2)[(x+3)^2]'}{[(x+3)^2]^2} = \frac{(x+3)^2(4x) - (2x^2-2)[2(x+3)1]}{(x+3)^4}$$

$$= \frac{(x+3)(4x) - 2(2x^2-2)}{(x+3)^3} = \frac{4x^2 + 12x - 4x^2 + 4}{(x+3)^3} = \frac{12x+4}{(x+3)^3},$$

and so in turn we have

$$f''(x) = \frac{(x+3)^3(12x+4)' - (12x+4)[(x+3)^3]'}{[(x+3)^3]^2} = \frac{12(x+3)^3 - (12x+4)[3(x+3)^2]}{(x+3)^6}$$

$$= \frac{12(x+3) - 3(12x+4)}{(x+3)^4} = \frac{12x + 36 - 36x - 12}{(x+3)^4} = \frac{-24x+24}{(x+3)^4} = \frac{-24(x-1)}{(x+3)^4}.$$

So we see that we have a critical point when $12x + 4 = 0$, i.e. when $x = \dfrac{-1}{3}$, and a potential inflection point when $-24(x-1) = 0$, i.e. when $x = 1$. And of course the function has a vertical asymptote at $x = -3$. Since these points are important, let's find out exactly where they're going to be plotted on our graph. We already know that $f(1) = 0$ from above, and we can calculate that

$$f(-1/3) = \frac{2(-1/3)^2 - 2}{(-1/3+3)^2} = \frac{-16/9}{64/9} = \frac{-16}{64} = \frac{-1}{4},$$

Now let's check what happens in between:

	$(-\infty, -3)$	$(-3, -1/3)$	$(-1/3, 1)$	$(1, \infty)$
f'	+	-	+	+
f''	+	+	+	-

I filled in this table by plugging in

$$f'(-4) = \frac{12(-4)+4}{(-4+3)^3} = \frac{-44}{-1} = 44 > 0, \qquad f''(-4) = \frac{-24(-4-1)}{(-4+3)^4} = \frac{120}{1} = 120 > 0,$$

$$f'(-2) = \frac{12(-2)+4}{(-2+3)^3} = \frac{-20}{1} = -20 < 0, \qquad f''(-2) = \frac{-24(-2-1)}{(-2+3)^4} = \frac{72}{1} = 72 > 0,$$

$$f'(0) = \frac{12(0)+4}{(0+3)^3} = \frac{4}{27} > 0, \qquad f''(2) = \frac{-24(2-1)}{(2+3)^4} = \frac{-24}{625} < 0.$$

Of course you can use different points if they look easier to plug in. Putting together everything we've seen so far, we want to sketch a function such that

- The graph goes through the point $(-1, 0)$
- There is a local minimum at $(-1/3, -1/4)$ and an inflection point at $(1, 0)$.
- The function approaches $y = 2$ asymptotically to the left and to the right.
- The function has a vertical asymptote $x = -3$.
- The function is decreasing on $(-3, -1/3)$ and increasing everywhere else.
- The function is convex up on $(-\infty, 1)$ and convex down on $(1, \infty)$.